LAPLACE EXPANSION OF DETERMINANTS

Definition 1 (Determinant)**.** *The determinant is the unique function*

$$\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ copies}} \to \mathbb{R}$$

satisfying the following properties:

(D1) (Multi-linear) The det *function is linear in each component separately, that is, for any vectors* $\mathbf{v_1}, \ldots, \mathbf{v_n}, \mathbf{w}$ *in* $\mathbb{R}^n$*, for any positive integer* k *with* $1 \le k \le n$*, and for any scalar* $\lambda \in \mathbb{R}$*,*

$$\det(\mathbf{v_1}, \ldots, \mathbf{v_k} + \lambda\mathbf{w}, \ldots, \mathbf{v_n}) = \det(\mathbf{v_1}, \ldots, \mathbf{v_k}, \ldots, \mathbf{v_n}) + \lambda \det(\mathbf{v_1}, \ldots, \mathbf{w}, \ldots, \mathbf{v_n}).$$

(D2) (Skew-symmetric) For each $1 \le i, j \le n$ *with* $i \ne j$*,*

$$\det(\mathbf{v_1}, \ldots, \mathbf{v_i}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_n}) = -\det(\mathbf{v_1}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_i}, \ldots, \mathbf{v_n})$$

(D3) (Normalization) $\det(\mathbf{e_1}, \ldots, \mathbf{e_n}) = 1$*, where* $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ *is the standard basis of* $\mathbb{R}^n$*.*

Definition 2 (Determinant of a square matrix)**.** *Let* V *be a* $n \times n$ *square matrix and let* v_j^i *be the* $(i, j)^{th}$ *entry of* A*. Denote the column vectors of* V *by*

$$\mathbf{v_j} = \begin{bmatrix} v_j^1 \\ v_j^2 \\ \vdots \\ v_j^n \end{bmatrix} \in \mathbb{R}^n,$$

$1 \le j \le n$*. Then the determinant of* V *is defined as*

$$\det(V) = \det(\mathbf{v_1}, \ldots, \mathbf{v_n}).$$

Theorem 3 (Laplace expansion of determinants)**.** *Let* V *be a* $n \times n$ *matrix with column vectors* $\mathbf{v_1}, \ldots, \mathbf{v_n}$*, where each* $\mathbf{v_j} \in \mathbb{R}^n$ *for* $1 \le j \le n$*. Then*

$$\det(V) = \sum_{i=1}^{n} (-1)^{i+j} v_j^i \det(M_{ij}),$$

where M_{ij} *is the* $(n-1) \times (n-1)$ *matrix obtained by deleting the* i^{th} *row and the* j^{th} *column of* V*.*

Proof. By the multi-linearity property *(D1)* of the determinants, for a fixed j, with $1 \le j \le n$,

$$\det(V) = \sum_{i=1}^{n} v_j^i \det(\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}, \mathbf{e_i}, \mathbf{v_{j+1}}, \ldots, \mathbf{v_n}). \tag{4}$$

For each i, with $1 \le i \le n$, we write the column vectors as a sum

$$\mathbf{v_k} = v_k^i \mathbf{e_i} + \widetilde{\mathbf{v}}_{\mathbf{k}}^i, \tag{5}$$

where the vector $\widetilde{\mathbf{v}}^i_{\mathbf{k}}$ is given by

$$\widetilde{\mathbf{v}}^i_{\mathbf{k}} = \begin{bmatrix} v_k^1 \\ \vdots \\ v_k^{i-1} \\ 0 \\ v_k^{i+1} \\ \vdots \\ v_k^n \end{bmatrix}$$

for $1 \leq k \leq n$, $k \neq j$. By skew-symmetry property *(D2)*, the determinant of linearly dependent vectors is zero. So substituting (5) to each summand of (4), we get

$$\det(V) = \sum_{i=1}^{n} v_j^i \det(\widetilde{\mathbf{v}}^i_{\mathbf{1}}, \ldots, \widetilde{\mathbf{v}}^i_{\mathbf{j-1}}, \mathbf{e_i}, \widetilde{\mathbf{v}}^i_{\mathbf{j+1}}, \ldots, \widetilde{\mathbf{v}}^i_{\mathbf{n}}). \tag{6}$$

We now define vectors $\hat{\mathbf{v}}^i_{\mathbf{k}} \in \mathbb{R}^{n-1}$ by removing the i^{th} component of vectors $\widetilde{\mathbf{v}}^i_{\mathbf{k}}$

$$\hat{\mathbf{v}}^i_{\mathbf{k}} = \begin{bmatrix} v_k^1 \\ \vdots \\ v_k^{i-1} \\ v_k^{i+1} \\ \vdots \\ v_k^n \end{bmatrix} \in \mathbb{R}^{n-1}.$$

Note that the matrix M_{ij} is given by

$$M_{ij} = \left[\hat{\mathbf{v}}^i_{\mathbf{1}}, \ldots, \hat{\mathbf{v}}^i_{\mathbf{j-1}}, \hat{\mathbf{v}}^i_{\mathbf{j+1}}, \hat{\mathbf{v}}^i_{\mathbf{n}}\right]. \tag{7}$$

From (6) and (7), it suffices to show that

$$\det(\widetilde{\mathbf{v}}^i_{\mathbf{1}}, \ldots, \widetilde{\mathbf{v}}^i_{\mathbf{j-1}}, \mathbf{e_i}, \widetilde{\mathbf{v}}^i_{\mathbf{j+1}}, \ldots, \widetilde{\mathbf{v}}^i_{\mathbf{n}}) = (-1)^{i+j} \det(M_{ij}).$$

If $\mathbf{u} \in \mathbb{R}^{n-1}$ is a column vector then define $\check{\mathbf{u}}^i \in \mathbb{R}^n$ by adding a 0 between the $(i-1)^{th}$ and the i^{th} entry of the vector $\mathbf{u}$, that is,

$$\check{\mathbf{u}}^i = \begin{bmatrix} u^1 \\ \vdots \\ u^{i-1} \\ 0 \\ u^i \\ \vdots \\ u^{n-1} \end{bmatrix} \in \mathbb{R}^n, \text{ where } \mathbf{u} = \begin{bmatrix} u^1 \\ \vdots \\ u^{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}.$$

Note that if $\{\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}\}$ be the standard basis of $\mathbb{R}^{n-1}$. Then

$$\check{\mathbf{f}}^i_{\mathbf{k}} = \begin{cases} \mathbf{e_k} & , k < i \\ \mathbf{e_{k+1}} & , k \geq i+1 \end{cases}.$$

Now consider the function

$$\Delta : \underbrace{\mathbb{R}^{n-1} \times \cdots \times \mathbb{R}^{n-1}}_{\text{n-1 copies}} \to \mathbb{R}$$

defined as

$$\Delta(\mathbf{u_1}, \ldots, \mathbf{u_{n-1}}) = (-1)^{i+j} \det(\check{\mathbf{u}}^i_{\mathbf{1}}, \ldots, \check{\mathbf{u}}^i_{\mathbf{j-1}}, \mathbf{e_j}, \check{\mathbf{u}}^i_{\mathbf{j+1}}, \ldots, \check{\mathbf{u}}^i_{\mathbf{n-1}}).$$

Then note that

$$\Delta(\mathbf{\hat{v}_1^i}, \ldots, \mathbf{\hat{v}_{j-1}^i}, \mathbf{\hat{v}_{j+1}^i}, \mathbf{\hat{v}_n^i}) = (-1)^{i+j} \det(\mathbf{\widetilde{v}_1^i}, \ldots, \mathbf{\widetilde{v}_{j-1}^i}, \mathbf{e_i}, \mathbf{\widetilde{v}_{j+1}^i}, \ldots, \mathbf{\widetilde{v}_n^i}).$$

Clearly Δ is multi-linear and skew-symmetric because of the properties *(D1)* and *(D2)* of the determinant. To show that $\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) = 1$, we look at the following cases:

Case 1: If $i = j = 1$ or $i = j = n$. Then

$$\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) = (-1)^{i+j} \det(\mathbf{e_1}, \ldots, \mathbf{e_n}) = (-1)^{i+j} = 1.$$

Case 2: If $i = 1$ and $1 < j < n$. Then

$$\begin{aligned}\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) &= (-1)^{i+j} \det(\mathbf{e_2}, \ldots, \mathbf{e_j}, \mathbf{e_1}, \mathbf{e_{j+1}}, \ldots, \mathbf{e_n}) \\ &= (-1)^{1+j}(-1)^{j-1} \det(\mathbf{e_1}, \ldots, \mathbf{e_n}) \\ &= 1.\end{aligned}$$

Case 3: If $j = n$ and $1 < i < n$. Then

$$\begin{aligned}\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) &= (-1)^{i+j} \det(\mathbf{e_1}, \ldots, \mathbf{e_{i-1}}, \mathbf{e_{i+1}}, \ldots, \mathbf{e_n}, \mathbf{e_i}) \\ &= (-1)^{i+n}(-1)^{n-i} \det(\mathbf{e_1}, \ldots, \mathbf{e_n}) \\ &= 1.\end{aligned}$$

Case 4: If $1 < i < j < n$. Then

$$\begin{aligned}\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) &= (-1)^{i+j} \det(\mathbf{e_1}, \ldots, \mathbf{e_{i-1}}, \mathbf{e_{i+1}}, \ldots, \mathbf{e_j}, \mathbf{e_i}, \mathbf{e_{j+1}}, \ldots, \mathbf{e_n}) \\ &= (-1)^{i+j}(-1)^{j-i} \det(\mathbf{e_1}, \ldots, \mathbf{e_n}) \\ &= 1.\end{aligned}$$

Case 5: If $1 < j < i < n$. Then

$$\begin{aligned}\Delta(\mathbf{f_1}, \ldots, \mathbf{f_{n-1}}) &= (-1)^{i+j} \det(\mathbf{e_1}, \ldots, \mathbf{e_{j-1}}, \mathbf{e_i}, \mathbf{e_j}, \ldots, \mathbf{e_{i-1}}, \mathbf{e_{i+1}}, \ldots, \mathbf{e_n}) \\ &= (-1)^{i+j}(-1)^{i-j} \det(\mathbf{e_1}, \ldots, \mathbf{e_n}) \\ &= 1.\end{aligned}$$

Hence Δ is the determinant function and we get the desired Laplace expansion. $\square$