Tychonoff's Theorem

Theorem 1 (Tychonoff). *The product of any collection of compact topological spaces is compact with respect to the product topology.*

This article discusses a simple proof of Tychonoff's theorem, which is quite different from the original proof in [6]. A lot of machinery will be needed to prove this theorem but but everything is easy to follow. This is a good point to mention that the above theorem is equivalent to the axiom of choice (see [3]) and will require Zorn's lemma in its proof.

1 Nets

Many topological notions in a metric space can be rephrased in terms of sequences. For example limit points of a set, continuity of functions, compactness, etc. But there are topological spaces where sequences are not enough to produce such nice results.

Example 2. Let X be an uncountable set and let $x_0 \in X$. Define $\tau = \{U \subset X \ : \ x_o \notin U\} \cup \{U \subset X \ : \ X - U \text{ is countable}\}$. It is easy to check that τ is a topology on X (which is not first countable). One can also check that there is no sequence in $X - \{x_0\}$ that converges to x_0 in this topology.

The above example shows that there is no equivalent criterion of continuity of a function $f : X \to Y$ (Y is some other topological space) at x_0 in terms of sequences.

In 1992, Eliakim Moore and Herman Smith introduced the notion of nets in their paper [5] which generalizes the notion of sequences. But nets are not the only generalization of sequences. In 1937, Henri Cartan gave another generalization of sequences called filters. But for our purpose, we will only use nets.

Definition. A partially ordered set $(D, \leq)$ is called a **directed set** if each pair of elements have a common upper bound, i.e., given d_1 and d_2 in D, there exists $d \in D$ such that $d_1 \leq d$ and $d_2 \leq d$.

Example 3. Every non-empty totally ordered set is directed.

The next example is important will be used repeatedly in this article.

Example 4. Let X be a topological space and let $x \in X$. Consider the collection N_x of all (open) neighborhoods of x in X, i.e.,

$$N_x = \{U \subset X \ : \ x \in \text{int}(U)\},$$

where $\text{int}(U)$ is the interior of U. We partially order N_x by reverse inclusion, i.e., $U \leq V$ if and only if $V \subset U$. It is easy to verify that $(N_x, \leq)$ is indeed a partially ordered set. Also, if $U \in N_x$ and $V \in N_x$ are neighborhoods of x, then $U \cap V \in N_x$ and $U \leq U \cap V$ and $V \leq U \cap V$. Hence $(N_x, \leq)$ is a directed set.

Recall that a sequence in a set X is a function from the set of natural numbers $\mathbb{N}$ to the set X. Note that $\mathbb{N}$ has a natural total order and hence it is a directed set.

Definition. A **net** in a set X is a pair (Φ, D) where D is a directed set and $\Phi : D \to X$ is some function.

It is easy to check that every sequence is a net, just take $D = \mathbb{N}$. The case when D is uncountable is more interesting. Whenever D is clear from the context, we will say that Φ is a net in X.

Definition. A net (Φ, D) in a topological space X is said to be **frequently** in a subset $A \subset X$ if for every $\alpha \in D$, there is a $\beta \in D$ such that $\beta \geq \alpha$ and $\Phi(\beta) \in A$.

Definition. A net (Φ, D) in a topological space X is said to be **eventually** in a subset $A \subset X$ if there is a $\beta \in D$ such that $\Phi(\alpha) \in A$ for every $\alpha \geq \beta$.

Proposition 5. *Let X be a topological space and let (Φ, D) be a net in X. If A and B are subsets of X, then*

1. *Φ is not frequently in A if and only Φ is eventually in $X \setminus A$.*
2. *Φ is eventually in A and Φ eventually in B if and only Φ is eventually in $A \cap B$.*

Proof. Trivial. ■

We are now interested in understanding convergence of nets.

Definition. A net (Φ, D) in a topological space X is said to converge to $x_0 \in X$ if Φ is eventually in every (open) neighborhood of x_0 in X.

Compare the above definition with that of a converging sequence.

Example 6. A net (Φ, D) in a discrete space X converges to $x \in X$ if and only if Φ is eventually in $\{x\}$, i.e., Φ is eventually constant.

Example 7. Any net (Φ, D) in an indiscrete space X converges to every point in X.

Example 8. Let X be a topological space and let $x \in X$. Let N_x be the directed set defined in Example 4, which is the set of all neighborhood of x with reverse inclusion order. For each $U \in N_x$, we pick $x_U \in U$. Then we define $\Phi : N_x \to X$ by $\Phi(U) = x_U$. By construction, this net Φ is eventually in any neighborhood of x and hence Φ converges to x.

Let us now see characterization of two topological objects using nets.

Proposition 9. *Let $f : X \to Y$ be a map between two topological spaces. Then f is continuous at $x_0 \in X$ if and only if for every net $\Phi : D \to X$ converging to x_0, the net $f \circ \Phi : D \to Y$ converges to $f(x_0) \in Y$.*

Proof. Suppose f is continuous at $x_0 \in X$ and let $\Phi : D \to X$ be a net in X converging to x_0. If $V \subset Y$ is an open neighborhood of $f(x_0)$, then by the continuity of f at x_0, there is an open neighborhood $U \subset X$ of x_0 such that $f(U) \subset V$. Since Φ is eventually in U, we get that $f \circ \Phi$ is eventually in $f(U) \subset V$. Hence $f \circ \Phi$ converges to $f(x_0)$.

Conversely, suppose that for every net Φ in X that converges to x_0, the net $f \circ \Phi$ converges to $f(x_0)$. To arrive at a contradiction, we assume that f is not continuous at x_0, i.e., there is a neighborhood V of $f(x_0)$ such that if U is a neighborhood of x in X, then $f(U) \not\subset V$. Hence for each neighborhood U of x, we can choose $x_U \in U$ such that $f(x_U) \notin V$. From Example 8, we know that the net $\Phi : N_x \to X$ defined by $\Phi(U) = x_U$ converges to x. Hence by hypothesis, $f \circ \Phi$ converges to $f(x_0)$, i.e., th net $f \circ \Phi$ is eventually in every neighborhood of $f(x_0)$, so in particular in V. But this is a contradiction because $f(x_U) \notin V$ for every $U \in N_x$. Hence f is continuous at x_0. ■

Proposition 10. *If A is a subset of a topological space X then the closure $\overline{A}$ of A in X coincides with the set of limits of nets in A that converges in X.*

Proof. Let $\mathscr{L}_A$ be the collection of all limit points of nets in A, i.e., $\mathscr{L}_A = \{x \in X$: there is a net $\Phi : D \to A$ converging to $x_0\}$. We want to show $\overline{A} = \mathscr{L}_A$. Let $x \in \overline{A}$. Then any neighborhood U of x intersects A, i.e., for every open neighborhood U of x, there is a point $a_U \in U \cap A$. From Example 8, we know that the net $\Phi : N_x \to A$ in A defined by $\Phi(U) = a_U$ converges to x. Hence $\overline{A} \subset \mathscr{L}_A$. Conversely, let $x \in \mathscr{L}_A$ and let $\Phi : D \to A$ be a net in A converging to x. Let U be a neighborhood of x. Then Φ is eventually in U, hence $U \cap A$ is non-empty. Hence $x \in \overline{A}$. ■

It is clear from Example 2 that we cannot replace the word "nets" in Proposition 10 by "sequences".

We have seen that a net in a topological space can converge to multiple points, like in Example 7. It is often convenient to deal with spaces where nets have unique limit of convergence.

Proposition 11. *A topological space X is Hausdorff if and only if each net (Φ, D) in X converges to at most one point.*

Proof. Suppose X is Hausdorff and and a net (Φ, D) in X converges to two distinct points x_1 and x_2. Since $x_1 \neq x_2$, there are disjoint open sets U_1 and U_2 such that $x_i \in U_i$ for $i = 1, 2$. Since Φ converges to x_i, it is eventually in U_i for $i = 1, 2$. This is a contradiction since $U_1 \cap U_2$ is empty.

Conversely, suppose each net in X converges to at most one point and X is not Hausdorff. Then there are distinct points x_1 and x_2 in X such that every neighborhood of x_1 intersects every neighborhood of x_2. Let N_i be the collection of open neighborhoods of x_i for $i = 1, 2$. Then we order the cartesian product $N_1 \times N_2$ by declaring $(U_1, U_2) \geq (V_1, V_2)$ if and only if $U_1 \subset V_1$ and $U_2 \subset V_2$. It is easy to verify that with this order, $N_1 \times N_2$ is a directed set. We know that for each $(U_1, U_2) \in N_1 \times N_2$, the intersection $U_1 \cap U_2$ is non-empty and hence we can pick $x_{U_1,U_2} \in U_1 \cap U_2$. If $(U_1, U_2) \geq (V_1, V_2)$, then $x_{U_1,U_2} \in U_1 \cap U_2 \subset V_1 \cap V_2$. Then the net $\Phi : N_1 \times N_2 \to X$ defined by $\Phi(U_1, U_2) = x_{U_1,U_2}$ converges to both x_1 and x_2, which is a contradiction. ■

2 Subnets

The notion of subnets is not as straightforward as subsequences. It is designed to allow us to generalize the results about subsequences.

Definition. A function $h : D' \to D$ between two directed sets D' and D is called **final** if for each $\beta \in D$ there is $\alpha_0 \in D'$ such that whenever $\alpha \geq \alpha_0$ in D', we have $h(\alpha) \geq \beta$ in D.

Definition. A net (Ψ, D') in a topological space X is a **subnet** of a net (Φ, D) in X if there is a final function $h : D' \to D$ such that $\Psi(\alpha) = \Phi(h(\alpha))$ for each $\alpha \in D'$.

Proposition 12. *A net (Φ, D) is frequently in each neighborhood of $x \in X$ if and only if there is a subnet of Φ that converges to x.*

Proof. Suppose (Φ, D) is frequently in every neighborhood of $x \in X$. Let N_x be the collection of all neighborhood of x. Define $D' = \{(\alpha, U) \in D \times N_x \ \ \Phi(\alpha) \in U\}$ and order D' by declaring $(\alpha, U) \leq (\beta, V)$ if and only if $\alpha \leq \beta$ and $U \supset V$. Then it is easy to verify that $(D', \leq)$ is a directed set. We claim that that the function $h : D' \to D$ given by $h(\alpha, U) = \alpha$ is final. Let $\beta \in D$. If $U_0 \in N_x$, then Φ is frequently in U_0 and hence there exists $\alpha_0 \geq \beta$ in D such that $\Phi(\alpha_0) \in U_0$. Hence $(\alpha_0, U_0) \in D'$. For this $(\alpha_0, U_0) \in D'$, we get that $(\alpha, U) \geq (\alpha_0, U_0) \implies h(\alpha, U) = \alpha \geq \alpha_0 \geq \beta$. Hence h is final. We now claim that the subnet $(\Phi \circ h, D')$ of (Φ, D) converges to x. If $V \in N_x$, then there exists $\beta \in D$ such that $\Phi(\beta) \in V$, hence $(\beta, V) \in D'$. So whenever $(\alpha, U) \geq (\beta, V)$, we have $(\Phi \circ h)(\alpha, U) = \Phi(\alpha) \in U \subset V$. Hence $\Phi \circ h$ converges to x.

Conversely, suppose subnet $(\Phi \circ h, D')$ of (Φ, D) converges to x. Let $\alpha \in D$ and let $U \in N_x$ (as defined above). We know that $\Phi \circ h$ is eventually in U, so there exists $\beta' \in D'$ such that whenever $\gamma' \geq \beta'$, we have $\Phi(h(\gamma')) \in U$. Since h is final, there exists $\beta'' \in D'$ such that whenever $\gamma'' \geq \beta''$, we have $h(\gamma'') \geq \alpha$. Let $\beta^* \in D'$ be such that $\beta^* \geq \beta'$ and $\beta^* \geq \beta''$. Let $\beta = h(\beta^*) \in D$. Then we have $h(\beta^*) = \beta \geq \alpha$ and $\Phi(h(\beta^*)) = \Phi(\beta) \in U$. So Φ is frequently in U. ■

The next concept for nets that we will discuss is very powerful and has no analogue for sequences.

Definition. A net (Φ, D) in X is called **universal** if for any subset $A \subset X$, either Φ is eventually in A or Φ is eventually in $X \setminus A$.

The above definition may seem so strong that we may reasonably doubt the existence of such a thing. However:

Theorem 13. *Every net has a universal subnet.*

The above theorem is so powerful that it is equivalent to the axiom of choice. In fact, this is the part in the proof of Tychonoff's theorem where we use Zorn's Lemma (or Hausdorff Maximal Principle). In order to make the proof of the above theorem easier, we introduce some new terminology which will only be used in the proof and nowhere else.

Definition. Let Φ be a net on X. Then a collection $\mathcal{C}$ of subsets of X is called **Φ-good** if the following two conditions are met:

1. If $U \in \mathcal{C}$ then Φ is frequently in U, and
2. if $U, V \in \mathcal{C}$, then $U \cap V \in \mathcal{C}$.

An example of a Φ-good collection would be $\mathcal{C} = \{X\}$. Let $\mathscr{C}_\Phi$ be the collection of all Φ-good collections of subsets of X. We order $\mathscr{C}_\Phi$ by defining

$$\mathcal{C} \leq \mathcal{C}' \iff \mathcal{C} \subset \mathcal{C}'.$$

Lemma 14. *If $\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$ is a linearly ordered subcollection of $\mathscr{C}_\Phi$ (with respect to the above defined partial order $\leq$), then*

1. $\mathcal{C}_\infty = \bigcup_{\lambda \in \Lambda} \{A : A \in \mathcal{C}_\lambda\}$ *is Φ-good, and*
2. $\mathcal{C}_\lambda \leq \mathcal{C}_\infty$ *for every $\lambda \in \Lambda$.*

Proof. If $U \in \mathcal{C}_\infty = \bigcup_{\lambda \in \Lambda} \mathcal{C}_\lambda$, then $U \in \mathcal{C}_\lambda$ for some $\lambda \in \Lambda$. Since $\mathcal{C}_\lambda$ is Φ-good, we get that Φ is frequently in U. Also if $U, V \in \mathcal{C}_\infty$, then $U \in \mathcal{C}_\lambda$ and $V \in \mathcal{C}_{\lambda'}$ for some $\lambda, \lambda' \in \Lambda$. Without loss of generality, we assume that $\mathcal{C}_\lambda \leq \mathcal{C}_{\lambda'}$, i.e., $U \in \mathcal{C}_{\lambda'}$. So we get $U \cap V \in \mathcal{C}_{\lambda'} \subset \mathcal{C}_\infty$. Hence $\mathcal{C}_\infty$ is Φ-good. The second statement follows trivially from the definition of $\mathcal{C}_\infty$ and $\leq$. ■

Proof of Theorem 13. Let $\Phi : D \to X$ be a net in X. From Zorn's lemma and Lemma 14, it follows that $\mathscr{C}_\Phi$ has a maximal element $\mathcal{C}_0$. Define $D_0 = \{(A, \alpha) : A \in \mathcal{C}_0, \alpha \in D,$ and $\Phi(\alpha) \in A\}$ and give a partial order on D_0 by defining $(A, \alpha) \leq (B, \beta)$ if and only if $B \subset A$ and $\alpha \leq \beta$. We claim that D_0 with this ordering is a directed set. We know that D is a directed set, given α and β in D, there exists $\gamma \in D$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. So if $(A, \alpha), (B\beta) \in D_0$, then $(A \cap B, \gamma) \in D_0$ (becasue $\mathcal{C}_0$ is Φ-good) and we have $(A, \alpha) \leq (A \cap B, \gamma)$ and $(B\beta) \leq (A \cap B, \gamma)$. Hence D_0 is a directed set.

Define $h : D_0 \to D$ by $h(A, \alpha) = \alpha$. We claim that h is a final function. Pick $\beta \in D$ and let $A_0 \in \mathcal{C}_0$ be arbitrary. Since A_0 is Φ-good, there exists $\alpha_0 \geq \beta$ such that $\Phi(\alpha_0) \in A_0$, so $(A_0, \alpha_0) \in D_0$. If $(A, \alpha) \geq (A_0, \alpha_0)$, then $h(A, \alpha) = \alpha \geq \alpha_0 \geq \beta$. Hence h is final.

This gives us a subnet $\Phi \circ h : D_0 \to X$ of Φ. We now claim that this subnet is universal. Let S be a subset of X and suppose $\Phi \circ h$ is frequently in S, i.e., given $(A, \alpha) \in D_0$, one can find $(B, \beta) \in D_0$ such that $(B, \beta) \geq (A, \alpha)$ and $(\Phi \circ h)(B, \beta) = \Phi(\beta) \in S$. By definition, $\Phi(\beta) \in B \subset A$. Hence $\Phi(\beta) \in A \cap S$. Hence Φ is frequently in $A \cap S$ for every $A \in \mathcal{C}_0$. By maximality of $\mathcal{C}_0$, we get that $S \in \mathcal{C}_0$ (otherwise, $\hat{\mathcal{C}}_0 = \mathcal{C}_0 \cup \{S\} \cup \{A \cap S : A \in \mathcal{C}_0\}$ is a Φ-good collection and $\mathcal{C}_0 \leq \hat{\mathcal{C}}_0$). If $\Phi \circ h$ is also frequently in $X \setminus S$, then by the same argument we get that $X \setminus S \in \mathcal{C}_0$. Then $\emptyset = S \cap (X \setminus S) \in \mathcal{C}_0$. This is a contradiction because Φ cannot be frequently in the empty set. So either $\Phi \circ h$ is not frequently in S or $\Phi \circ h$ is not frequently in $X \setminus S$. By Proposition 5, we get that either $\Phi \circ h$ is eventually in $X \setminus S$ or $\Phi \circ h$ is eventually in S. Hence $\Phi \circ h$ is a universal net. ■

3 Compactness

We now give a very useful characterizations of compactness in terms of nets, which generalizes the equivalence of sequential compactness and compactness in metric spaces.

Definition. A collection $\mathcal{F}$ of sets has the **finite intersection property** if the intersection of any finite subcollection of $\mathcal{F}$ is non-empty.

Theorem 15. *Let X be a topological space. Then the following are equivalent:*

1. *X is compact.*
2. *Every collection of closed subsets of X with the finite intersection property has a non-empty intersection.*

3. *Every net in X has a convergent subnet.*

4. *Every universal net in X converges.*

Proof. $\boxed{1 \Rightarrow 4}$ Suppose $\Phi : D \to X$ is a universal net which does not converge in X. So for each $x \in X$, there is a neighborhood U_x of x such that (Φ, D) is not eventually in U_x. Since (Φ, D) is universal, it is eventually $X \setminus U_x$. Since X is compact, the open cover $\{U_x : x \in X\}$ of X has a finite subcover, i.e., there exists $x_1, x_2, \ldots, x_k$ in X such that $X = \bigcup_{j=1}^{k} U_{x_j}$. Since (Φ, D) is eventually in $X \setminus U_{x_j}$ for each $j = 1, 2, \ldots, k$, we get that (Φ, D) is eventually in the intersection $\bigcap_{j=1}^{k} \left(X \setminus U_{x_j}\right) = X \setminus \bigcup_{j=1}^{k} U_{x_j} = \emptyset$. This is a contradiction and hence every universal net in a compact space X converges.

$\boxed{4 \Rightarrow 3}$ This follows from Theorem 13.

$\boxed{3 \Rightarrow 2}$ Let $\mathcal{F} = \{C\}$ be a collection of closed subsets of X having the finite intersection property. By adding finite intersections of elements of $\mathcal{F}$, we can assume that $\mathcal{F}$ is closed under finite intersection. We give $\mathcal{F}$ a partial order defined by $C \leq C' \iff C \supset C'$. It is easy to check that $(\mathcal{F}, \leq)$ forms a directed set. For each $C \in \mathcal{F}$, we pick $x_C \in C$. This defines a net $\Phi : \mathcal{F} \to X$ on X by $\Phi(C) = x_C$. By hypothesis, Φ has a convergent subnet given by a final map $h : D \to \mathcal{F}$. Thus for $\alpha \in D$, $h(\alpha) \in \mathcal{F}$, and $x_{h(\alpha)} \in h(\alpha)$. Suppose $x_{h(\alpha)} \to x$. Then we claim that $x \in \bigcap_{C \in \mathcal{F}} C$. Let $C \in \mathcal{F}$. Since h is final, there exists $\beta \in D$ such that $\alpha \geq \beta \implies h(\alpha) \subset C$ and hence $x_{h(\alpha)} \in h(\alpha) \subset C$. Since C is closed, Proposition 10 gives us that $x \in C$. Since C was arbitrary, we get $x \in \bigcap_{C \in \mathcal{F}} C$.

$\boxed{2 \Rightarrow 1}$ Let $\mathcal{O} = \{U\}$ be an open cover of X. To arrive at a contradiction, we suppose that X cannot be covered by a finite sub-collection of $\mathcal{O}$. Consider the collection of closed subsets of X given by $\mathcal{F} = \{X - U : U \in \mathcal{O}\}$. Then $\mathcal{F}$ has the finite intersection property since for for any finite sub-collection of $\mathcal{F}$, say $X - U_1, \ldots, X - U_k$, we have $\bigcap_{i=1}^{k}(X - U_i) = X - \bigcup_{i=1}^{k} U_i$ and by assumption $\bigcup_{i=1}^{k} U_i \neq X$. Hence by hypothesis $\mathcal{F}$ has non-empty intersection, i.e., $\emptyset \neq \bigcap_{U \in \mathcal{O}}(X - U) = X - \bigcup_{U \in \mathcal{O}} U = \emptyset$, a contradiction. Hence X is compact. ∎

4 Product topology

The product topology was also introduced by Tychonoff and it is also sometimes referred to as Tychonoff topology. The following property about nets on a product space will be used in the proof of Theorem 1.

Proposition 16. *A net (Φ, D) in a product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ of topological spaces $\{X_\alpha\}_{\alpha \in \Lambda}$ converges to $x \in X$ if and only if for each $\lambda \in \Lambda$, the net $\pi_\lambda \circ \Phi : D \to X_\lambda$ converges to $\pi_\lambda(x)$.*

Proof. Suppose that a net (Φ, D) in a product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ converges to $x \in X$. Then by definition of product topology, the projection $\pi_\lambda : X \to X_\lambda$ is continuous for each $\lambda \in \Lambda$. Hence by Proposition 9, we get $\pi_\lambda \circ \Phi : D \to X_\lambda$ converges to $\pi_\lambda(x) \in X_\lambda$.

Conversely, suppose that the net $(\pi_\lambda \circ \Phi, D)$ in X_λ converges to $\pi_\lambda(x)$ for each $\lambda \in \Lambda$. Let $U \subset X$ be an open neighborhood of x. Then there is a basic open set B such that $x \in B \subset U$. By the definition of basic open sets in the product topology, there exists $\lambda_1, \lambda_2, \ldots, \lambda_k$ in Λ and open sets $U_{\lambda_j} \subset X_{\lambda_j}$ for $j = 1, 2, \ldots, k$ such that $B = \pi_{\lambda_1}^{-1}(U_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(U_{\lambda_2}) \cap \cdots \cap \pi_{\lambda_k}^{-1}(U_{\lambda_k})$. By hypothesis, for each $j = 1, 2, \ldots, k$, there exists $d_j \in \Lambda$ such that $(\pi_{\lambda_j} \circ \Phi)(d) \in U_{\lambda_j}$ for all $d \geq d_j$ in Λ. Since D is a directed set, there exists $d^* \in \Lambda$ such that $d^* \geq d_j$ for all $j = 1, 2, \ldots, k$. So for all $d \geq d^*$ in λ we have $(\pi_{\lambda_j} \circ \Phi)(d) \in U_{\lambda_j}$ for all $j = 1, 2, \ldots, k$. Hence $\Phi(d) \in \bigcap_{j=1}^{k} \pi_{\lambda_j}^{-1}(U_{\lambda_j}) = B \subset U$. Hence the net (Φ, D) is eventually in U. Since U was an arbitrary neighborhood of x in X, we get that (Φ, D) converges to x. ∎

5 Proof of Theorem 1

We have all the tools needed to prove Theorem 1. We need one last lemma before we start the proof.

Lemma 17. *If* (Φ, D) *is a universal net in* Y *and* $\pi : Y \to Z$ *is any function, then* $(\pi \circ \Phi, D)$ *is a universal net in* Z.

Proof. Given $A \subset Z$, we have that Φ is eventually in $\pi^{-1}(A)$ or Φ is eventually in $Y \setminus \pi^{-1}(A)$. So $\pi \circ \Phi$ is eventually in A or $\pi \circ \Phi$ is eventually in $\pi(Y \setminus \pi^{-1}(A)) = \pi(\pi^{-1}(Z) \setminus \pi^{-1}(A)) = Z \setminus A$. ∎

Finally, we arrive at the proof of Tychonoff's theorem. If the proof looks too short, it is because we have done all the heavy work during the discussion about universal nets.

Proof of Theorem 1. Suppose $\{X_\alpha\}_{\alpha \in \Lambda}$ is a collection of compact topological spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$ be the product space. Let $\Phi : D \to X$ be a universal net in X. By Lemma 17, for each $\lambda \in \Lambda$, we get that $\pi_\lambda \circ \Phi : D \to X_\lambda$ is a universal net in X_λ and by Theorem 15, we get that $\pi_\lambda \circ \Phi$ converges to some $x_\lambda \in X_\lambda$. Then by Proposition 16, we get that Φ converges to $x = (x_\lambda)_{\lambda \in \Lambda} \in X$. Hence by Theorem 15, we get that X is compact. ∎

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